JOURNAL OF MATERIALS SCIENCE 7 (1972) 98-104

Viscous Flow of Aligned Composites

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The case is considered of an aligned composite subjected to tensile creep in the direction of the fibres. A geometrical argument shows that shear strain in the composite is amplified $l/2s$ times compared with unsupported matrix, where $l/2s \sim$ aspect ratio of the inter-fibre spaces. The shear stress is amplified $(l/2s)^{1/n}$ times, where n is the exponent in the matrix creep law. Consequently the rate of energy expenditure is amplified $V_m(l/2s)^{1+1/n}$ times, as is therefore the tensile flow resistance of the composite (V_m is the volume fraction of matrix). The potential increase in flow resistance is thus enormous. However, the fibre end-stress, which is calculated, $\propto$ fibre diameter, and may be large enough to initiate rupture unless the fibres are very thin (e.g. 1 μm diameter). The tensile load is roughly equally divided between matrix and fibres irrespective of volume fractions.

1. Introduction

There have been several analyses of the tensile creep of aligned fibre composites, the tensile stress being parallel to the fibre axis [1-4]. The most rigorous is that of Ferris [4], which because of its rigour is already quite complicated even though it is limited to matrices with simple linear viscosity. The most complete is that of Mileiko [3] who, like other authors, works in terms of stress. He presses simplicity to the point of using a wrong geometry that only allows extension of a sample to occur by creating holes; Mileiko recognised this but it did not seem to him important.

It is doubtful if any analysis in terms of stress can combine reasonable simplicity with conceptual accuracy, because the stress-state in the flowing matrix is quite complicated. This combination can however be achieved by making the analysis in terms of energy dissipated instead of in terms of stress. There is no gain in quantitative accuracy, but it seems to the writer that the physical picture obtained is clearer as well as being conceptually more correct, and some conclusions are more obvious.

Except for de Silva's treatment [2], which uses a perturbation of the elastic situation, none of the treatments is limited to creep; they apply also to hot flow as, e.g. during injection-moulding of filled polymers, subject to the restriction that they assume aligned fibres. These treatments, and also the present one, apply best when the plastic strain is substantially greater than the elastic, since they neglect elastic strains. At small plastic strains of the same magnitude as the elastic strain de Silva's treatment is probably the most nearly correct.

2. Composite Flow Stress

We can avoid discussion of the complicated stress system in the fibre-strengthened matrix by calculating in terms of the energy dissipated during flow. If a stress σ_c applied to unsupported matrix extends it at the speed $\dot{\epsilon}$, the rate at which work is done is $\sigma_0\dot{\epsilon}$ per unit volume. Similarly for the fibre-strengthened matrix the rate of doing work is $\sigma_0\dot{\epsilon}$ per unit volume, σ_c being the tensile stress required to produce extension at the rate $\dot{\epsilon}$. Let τ_0 and $\dot{\gamma}_0$ be the shear stress and shear strain rate in the unsupported matrix at 45° to the tensile axis; $\tau_0 = \frac{1}{2}\sigma_0$ and $\dot{\gamma}_0 = 2\dot{\epsilon}$. We have

$$\sigma_0\dot{\epsilon} = \tau_0\dot{\gamma}_0 \quad (1)$$

Let τ_c and $\dot{\gamma}_c$ be the significant shear stress and strain rate in the fibre-strengthened matrix; we have

$$\sigma_c\dot{\epsilon} = \tau_c\dot{\gamma}_c V_m \quad (2)$$

the term V_m (matrix volume fraction) being introduced because energy is assumed to be expended only in the matrix. The significant shear stress τ_c and shear strain γ_c are those parallel to the fibres because both these quantities, but especially the strain component, are greatly amplified as compared with τ_0 and γ_0 (e.g. equation 5 below), and consequently they are the cause of energy dissipation in the fibre-strengthened matrix. From geometrical considerations $\dot{\gamma}_c$ can be expressed in terms of $\dot{\gamma}_0$, and from knowledge of the matrix flow law τ_c can then be expressed in terms of τ_0. When this has been done σ_c can be expressed in terms of σ_0, to give therefore the amplification of flow resistance introduced by the fibre-strengthening.

The geometrical relations involved in the flow of fibre-strengthened material are simple, but different from those obtaining in the case when fibre and matrix undergo small, equal, elastic strains without large relative flow. Consider the element of fibre-strengthened solid, a piece of which is shown in fig. 1; under tensile loading this piece increases in length from p in fig. 1 to $p(1 + \epsilon)$ in fig. 2. The fibres themselves are

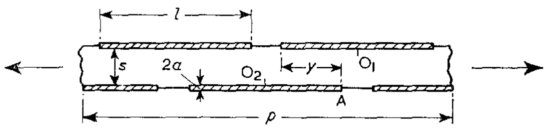

Figure 1 Element of fibre-strengthened composite loaded in tension as indicated by the arrows.

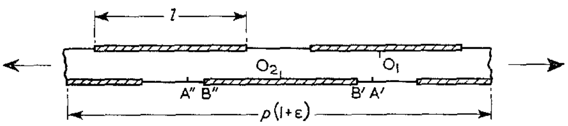

Figure 2 The element of fig. 1 after an extension ϵ.

supposed to be rigid and for the moment the matrix is supposed to extend uniformly, i.e. plane cross-sections remain plane. We make the assumption that the consequent relative motion between matrix and fibre is symmetrical about the centre-point of a fibre. The correctness of this assumption depends on the neighbouring fibres being distributed "uniformly", which raises a difficult problem of definition, but it seems the only sensible assumption to make. At the centre of a fibre, e.g. O_1 or O_2 (fig. 1), there is then zero relative motion, but at the fibre end a point in the matrix such as A (fig. 1) moves away from the end to A′ (fig. 2) and there is relative motion A′B′ (fig. 2) between fibre and matrix. At the opposite end there is relative motion A″B″ in the contrary sense. At any point initially distant x from the fibre centre there is relative motion

$$r = \epsilon x \tag{3a}$$

The mean value of r is

$$\bar{r} = \epsilon l/4, \tag{3b}$$

where l is the fibre length. Since the fibres are rigid, any point in a fibre will serve as reference-point from which to measure its relative motion, but the matrix does not extend uniformly as supposed in drawing fig. 2 unless at the interface between matrix and fibre there is completely uninhibited sliding. A more realistic supposition is the opposite one, namely that sliding is negligible. A decision must then be made about which point in the matrix to choose as reference point. In so far as two adjacent fibres such as the two depicted in fig. 3 can be considered in isolation, which again depends on the fibres being distributed uniformly, the centre line between them, e.g. YY′ in fig. 3, is symmetrically

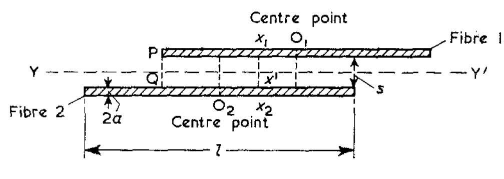

Figure 3

disposed with respect to both fibres and therefore contains reference points in the overlap region which are suitable because they give the same answer for either fibre. Thus, for any point X_1 in fibre 1 of fig. 3 we choose as matrix reference the point X^1 opposite X_1 and lying on the centre line; the corresponding point in fibre 2 is X_2. The relative motion r in equation 3 then involves a shear strain in the matrix of magnitude $\gamma_c = 2r/s$ where s is the spacing between fibres. Substituting in equation 3b gives the average matrix shear strain γ_c in the fibre-strengthened material as

$$\gamma_c = \epsilon l/2s \tag{4}$$

However, equation 4 relates to the shear depicted schematically in fig. 4, where the dashed lines indicate the shear—PQ′Q in fig. 4 represents the sheared state of PQ in fig. 3. Since PQ′Q is not perpendicular to the fibres as PQ is, the pieces of matrix in adjoining units like that in fig. 4 no longer fit together; there are holes in the matrix similar to Mileiko's holes. In order to fill up the holes additional shear must take place.

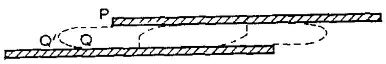

Figure 4 Depicting part of the shear that occurs in the matrix. PQ in fig. 3 becomes PQ′Q here. There must, however, be additional shear if a hole is not to form to the left of PQ′Q.

It is probably not far wrong to assume a doubling of the shear given in equation 4.

We are now able to calculate the amplification of shear strain and hence of shear stress in the fibre-strengthened matrix. In a separate block of matrix the shear strain accompanying an extension ϵ is $\gamma_0 = 2\epsilon$. Comparison with equation 4 shows that in the fibre-strengthened material the shear strain is magnified by the factor

$$\frac{\gamma_c}{\gamma_0} = l/2s \tag{5}$$

when allowance is made for the doubling referred to. This shear magnification must clearly also apply to the shear strain *rates*. From this amplification of shear strain rate the amplification of shear stress can be obtained by using the flow law, which often approximates to a power law connecting shear stress τ with strain rate $\dot{\gamma}$:

$$\dot{\gamma} = \alpha\tau^n \tag{6}$$

where α and n are experimentally determined constants. We therefore have

$$\tau_c = \left(\frac{\dot{\gamma}_c}{\alpha}\right)^{1/n} = \left(\frac{l\dot{\gamma}_0}{2s\alpha}\right)^{1/n} = \left(\frac{l}{2s}\right)^{1/n}\tau_0 \tag{7}$$

using equations 5 and 6 and applying the ratio in equation 5 to the strain rates. The shear stress in the composite is consequently magnified by the factor

$$\tau_c/\tau_0 = \left(\frac{l}{2s}\right)^{1/n} \tag{8}$$

by comparison with unsupported matrix extending at the same rate.

Inserting in equation 2 the amplification factors for strain and stress, i.e. substituting for τ_c from equation 8 and for $\dot{\gamma}_c$ from equation 5

$$\sigma_c = V_m\left(\frac{l}{2s}\right)^{1+1/n}\sigma_0 \tag{9}$$

Thus the stress σ_c required to extend the fibre-strengthened matrix at a given rate is magnified $V_m(l/2s)^{1+1/n}$ times compared with the stress σ_0 that extends the unsupported matrix at the same rate. The magnification arises mainly because of the shear strain (and shear strain rate) amplification in the fibre-strengthened matrix (equation 5) but partly also because the shear stress is amplified, though to a lesser degree (equation 8).

Equation 9 cannot be exact because it states that σ_c is zero when $l/2s$ is zero, i.e. in unsupported matrix, whereas the correct value is then $\sigma_c = \sigma_0$. The reason for this error lies in the approximate nature of the "doubling assumption", made to avoid holes, in the calculation leading to equation 5. An exact assumption would produce an additional $V_m\sigma_0$ term in equation 9. An apparently simple way of making the correction is to assume that the extra shear strain needed to avoid holes occurs as it would in unsupported matrix, i.e. without being amplified by the presence of the fibres. With this assumption a $V_m\sigma_0$ term is correctly introduced. But it does not seem at all clear that this assumption is correct. In any case, in composites the correction is slight since $\sigma_c \gg \sigma_0$, and is almost certainly a far less important consideration than the question of irregular fibre distribution referred to in the discussion. The correction is therefore neglected.

It has been assumed in fig. 4 (and equation 4 onwards) that no slip occurs at the interface between matrix and fibre. Whether slip does or does not occur is an interesting scientific question. In hydrodynamic problems concerned with fluids of non-zero viscosity flowing past solid bodies it is accepted that there is no sliding at the interface itself, i.e. that the same interface atoms of fluid and body remain linked together [5, 6]. On the other hand, in polycrystals undergoing creep, it is known that there is much sliding at grain boundaries even though they are an unusually cohesive type of interface. Probably the difference between the two situations arises because far greater stresses are needed to deform polycrystals, even when hot, than to cause flow in a liquid like water, and the much greater stress is clearly more likely to cause sliding at the interface. Although, therefore, we do not yet know whether interface sliding in any hot composite actually occurs and only experiment can determine whether it does, to reckon with it is only reasonable. Its effect would be to reduce the stress σ_c (equation 9), since some of the shear strain will be replaced by sliding; in fig. 2, for instance, all the strain is so replaced. In the limiting situation, when at the interface there is no resistance at all to sliding, the matrix deforms as does unsupported matrix; there is no shear magnification and $\sigma_c = V_m\sigma_0$ where V_m is the volume fraction of matrix.

It has also been assumed that the fibres contribute no Orowan hardening to the matrix nor interfere with recovery. As Street [7] has pointed out, in soft crystalline matrices filled with closely spaced fibres (such as rod eutectics in which the rod spacing may be $\sim 1\ \mu$m) the fibres should impede slip quite strongly; they should also retard recovery and in both these ways make a contribution to deformation resistance additional to the contributions already discussed. There is presumably a similar additional impedance contribution with polymer matrices that deform by shear concentrated on one plane, like slip in crystals.

3. Distribution of Stress

The tensile stress σ_f in the fibre at any point is obtained in the usual way by integrating interface shear stress τ from the fibre end. Thus, assuming cylindrical fibres of radius a and length l,

$$\sigma_f \pi a^2 = -2\pi a \int_{l/2}^{x} \tau \mathrm{d}x \tag{10}$$

The rate law of equation 6 enables the stress τ to be defined in terms of the rate $\dot{r}$ of relative motion r (equation 3a) as $\beta\tau^n = \dot{r}$, where β is the appropriate proportionality factor. Hence, using equation 3a to substitute for $\dot{r}$

$$\sigma_f = -\frac{2}{a}\int_{l/2}^{x}\left(\frac{\dot{\epsilon}x}{\beta}\right)^{1/n}\mathrm{d}x$$

$$= \frac{2n}{a(1+n)}\left(\frac{\dot{\epsilon}}{\beta}\right)^{1/n}\left[\left(\frac{l}{2}\right)^{1+1/n} - x^{1+1/n}\right] \tag{11}$$

which gives the relation between σ_f and x shown in fig. 5 for several values of n. The interesting point about the relation is that for high values of n, which apply to some practical matrices, the tensile stress in the fibre rises nearly linearly from the fibre-end to the fibre-centre, a point that was noted by Kelly and Tyson [1].

The tensile force in the matrix is also given by an expression like that in equation 11, but obtained by integrating from a position opposite the fibre-centre to a position opposite the fibre-end. For high values of n this force also rises nearly linearly over the whole distance from the fibre-centre to the fibre-end. Assuming that the tensile force in the matrix is zero at a point opposite the fibre-centre we then have

mean tensile force supported by fibres $V_f\,\bar{\sigma}_f$
$\sim$ mean tensile force supported by matrix $V_m\,\bar{\sigma}_m$ ..... (12)

i.e. matrix and fibre bear nearly equal shares of

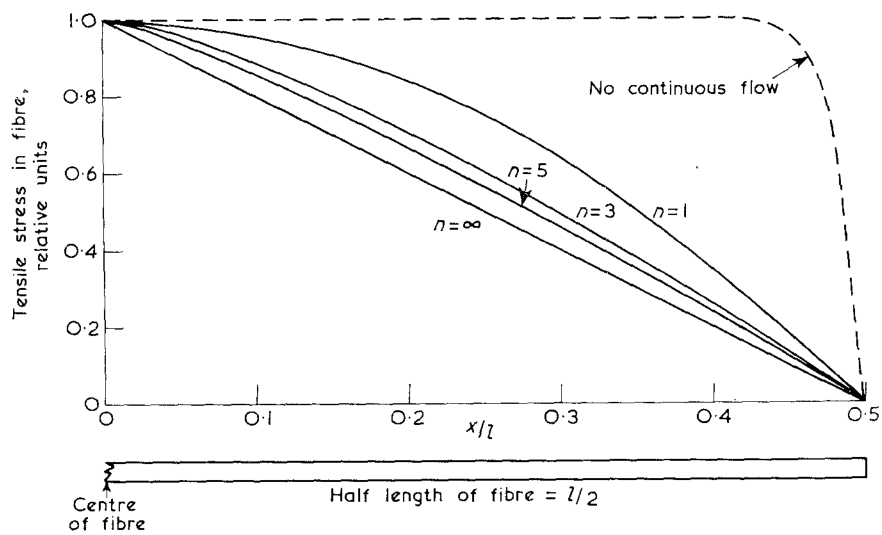

Figure 5 Variation of tensile stress along the length of a fibre. The full line curves apply when flow occurs continuously with time and describe the variation of stress for different values of n in the matrix flow law of equation 6. This is the situation discussed in this paper. The dashed curve applies when the matrix flow *rate* is zero, i.e. when a fixed elastic or plastic strain occurs.

the total applied force. This conclusion can also be seen in Ferris' detailed calculation [4]. The load sharing is quite different from that in the case usually treated where no continuous flow occurs and most of the load is borne by the fibres, essentially because the force distribution along them is then of the type depicted by the dashed curve in fig. 5. The force distribution corresponding to the solid lines in fig. 5 precludes any marked preferential loading of the fibres. Since the matrix nevertheless does not extend rapidly the shear stress at 45° to the axis, which is generated by the axial tensile stress in the matrix and causes extension, cannot be large. Consequently there must exist a lateral tensile stress in the matrix which offsets most of this shear stress and which consequently may be large, and which helps to create the complicated non-uniform multi-axial stress state in the matrix referred to earlier.

4. End Force on Fibres

There is also an end stress on each fibre. According to equation 3a at the end of each fibre there is relative motion between fibre and matrix at the speed

$$\dot{r} = \dot{\epsilon} l/2 \qquad (13)$$

As long as no cavity opens up at the fibre end, there must trail in the wake of the fibre as it moves relative to the matrix the thin pyramid of matrix material AZ shown in fig. 6, which increases the effective length of the fibres from l to $l + 2z$ (counting the pyramids at both ends),

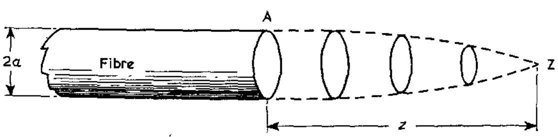

Figure 6 As long as a cavity does not open up at a fibre end, a pyramid AZ of matrix must move with the fibre relative to the surrounding matrix. The movement can be represented by means of the dislocation loops shown, which travel to the right, from A to Z, as the pyramid moves to the left relative to adjoining matrix.

where z is the length AZ of a pyramid. Hence, the end stress σ_e at A can be computed by employing the standard procedure used in deriving equation 11 for σ_f; i.e. by integrating, from Z to A, the shear stress applied to the pyramid of matrix. Then, if σ_f' is the fibre-centre stress,

$$\sigma_e/\sigma_f' \sim 2z/l \qquad (14)$$

If we can find another expression connecting σ_e and z, both are known. Another expression can be obtained by imagining the tensile strain of magnitude σ_e/E_m (E_m is the matrix Young's modulus) necessarily present in the pyramid as existing in the form of the n dislocation loops, each of thickness (Burgers vector) $\mathbf{b}$, indicated in fig. 6. Then

$$n\mathbf{b} = \tfrac{1}{2}\sigma_e\, z/E_m \qquad (15)$$

the factor $\frac{1}{2}$ arising because the tensile stress is assumed to build up linearly from Z to A. Equation 15 is not the desired expression because it contains two new unknowns, n and $\mathbf{b}$, which however can be eliminated. Each loop disappears by diffusion in a calculable time t' (see appendix) given, for cylindrical fibres, by

$$t' = \frac{\pi\, k\, T\, a^2}{2D\, G\, \mathbf{b}^3} = \frac{3\pi^2\, a^2\, \eta}{G\, \mathbf{b}^2} \qquad (16)$$

where D is the diffusion coefficient, η the viscosity, G the shear modulus, k is Boltzmann's constant and T the absolute temperature. In time t' the relative displacement between fibre end A and surrounding matrix is $\dot{r}t'$ and this must be equal to the tensile strain given by equation 15. Hence, using equations 13 and 15

$$\dot{\epsilon}\, l\, t'/2 = \dot{r}\, t' = n\, \mathbf{b} = \sigma_e\, z/2E_m \qquad (17)$$

Equation 17 is the desired expression since the new parameters in it are known – t' – from equation 16 and E_m as the Young's modulus of the matrix. Combining equations 14 and 17,

$$\left.\begin{aligned} \sigma_e &= \sqrt{2\sigma'_f\, \dot{\epsilon}\, t'\, E_m} \\ z &= \sqrt{l^2\, \dot{\epsilon}\, t'\, E_m/2\sigma_f'} \end{aligned}\right\} \qquad (18)$$

If the matrix is aluminium at 300°C and $a = 1$ μm (i.e. appropriate to a rod eutectic), $t' \sim 4$ sec. Assuming that $\sigma_f' = 10^7$ Pa (1 kgf/mm^2), $\dot{\epsilon} = 10^{-7}$/sec, and $l = 10$ mm, then $\sigma_o = 7 \times 10^5$ Pa and $z = 0.5$ mm. In this case both the end stress and the effective extra length $\sim 2z$ which the pyramids add on to the fibres can be neglected. However, by substituting in equation 18 for t' from equation 16 it can be seen that $\sigma_e \propto$ fibre radius a, so that fat fibres are undesirable as the end stresses may be large enough to provoke rupture. In the case of a hot polymer "filled" with glass fibres flowing at the speeds enforced during extrusion-moulding, both σ_e and z are likely to be far from negligible, in which case the end effect contributes considerably to the flow resistance.

5. Discussion

Equation 9 expresses the flow resistance of a fibre-strengthened matrix in terms of the physically significant parameters (and – to remind the reader – applies best when the elastic strain can be neglected, as do other treatments). Thus, the parameter $l/2s$, i.e. fibre length divided by twice the fibre spacing or, approximately, the aspect ratio y/s of the inter-fibre spaces (see fig. 1) should be large. V_m should also be maximised by using thin fibres, which implies a large fibre aspect ratio. It is, however, usual to express fibre-strengthening in terms of fibre aspect ratio ρ and volume fractions V_f and V_m. Since l/s equals $\rho V_f/V_m$ for lamellar fibres and approximately equals $\rho V_f/(\sqrt{\pi}\, V_f - 2V_f)$ for cylindrical fibres, equation 9 is equivalent to

$$\left.\begin{aligned} &\text{lamellar fibres} \quad \sigma_c = \frac{\rho^{1+1/n}}{2^{1+1/n}} \frac{V_f^{1+1/n}}{V_m^{1/n}} \sigma_0 \\ &\text{cylindrical fibres} \\ &\sigma_c = \frac{\rho^{1+1/n}}{2^{1+1/n}} \frac{V_f^{1+1/n}\, V_m}{(\sqrt{\pi V_f} - 2V_f)^{1+1/n}} \sigma_0 \end{aligned}\right\} \quad (19)$$

σ_0 being the stress that produces the same extension rate in the unsupported matrix as σ_c does in the fibre-strengthened matrix. Mileiko's corresponding equation 17 (which can be expressed as

$$\sigma_c = \frac{\rho^{1+1/n}}{2^{2+1/n}} \frac{V_f^{1+1/n}}{V_m^{1/n}} \sigma_0$$

for lamellar fibres in the present notation) is similar to equation 19. As Mileiko shows, his equation fits experimental data approximately, so equations 9 or 19 will as well. Because the flow resistance may be greatly reduced by interface-sliding or increased by interference with slip and recovery, there is no point in taking comparisons further until we have information about these factors. However, as an example of the theoretical strengthening available, assume 0.5 volume fraction of fibres with aspect ratio 100 in a matrix with $n = 5$ (e,g. Al or Ni) when, according to equation 19, the fibre-strengthened matrix will carry 96 times as large a stress as the unsupported matrix for a given rate of extension, or (equation 6) at the same applied stress will extend 8×10^9 times more slowly. This degree of strengthening of a metal matrix exceeds the best obtainable by conventional alloying.

Since interface sliding reduces the effective fibre-reinforcement, in the limit to zero, the question whether it does or does not occur is important. Although in a composite intended for creep resistance sliding is undesirable, in the extrusion moulding of glass-filled polymers interface layers that assist sliding are beneficial provided the finished room temperature properties are not harmed.

The sensitivity of a fibre-strengthened matrix to fibre distribution is worth pointing out. By combining equation 9 with the strain rate law of equation 6 it follows that $\dot{\epsilon} \propto (2s/l)^{n+1}$. This relationship applies to any local volume element large enough to contain several fibres. The local strain rate is then highly sensitive to variations in (l/s), i.e. to fibre distribution. In a fibre-strengthened matrix designed for creep resistance the useful life, namely the rupture life, is probably determined by local strain, e.g. through a strain-dependent cavitation process. Consequently, the rupture-life is likely to be heavily reduced by inhomogeneities in fibre distribution, although the overall strain rate is little affected.

When the matrix is a metal and the temperature low, the amplified shear strain described by equation 5 will manifest itself as rapid strain-hardening in a tensile test, and thus helps to account for the observation of rapid strain-hardening which appears to have been made in a discontinuous fibre-composite by Lee [8] and which characterises pearlite.

6. Conclusions

1. The composite's flow resistance is mainly due to the fact that, in comparison with unsupported matrix, the shear strain is greatly amplified for a given overall extension; in addition, the shear stress required is necessarily also larger in the composite. As far as metal matrices are concerned, the theory indicates that a flow resistance superior to that obtained by conventional alloying is achievable in principle.
2. The end stress on a fibre $\propto$ fibre diameter (cylindrical fibres). Thus for creep service, in so far as rupture is initiated at fibre ends, fine fibres are better than fat fibres. Moreover, rupture-life is expected to be shortened drastically by inhomogeneous distribution of the fibres.
3. As long as flow in the matrix occurs, the applied tensile *load* is approximately equally divided between fibres and matrix independently of relative volume fractions.

Acknowledgements

Discussion with Dr S. T. Mileiko and Dr K. N. Street has been a continual stimulus. The work was done as part of the general research programme of the National Physical Laboratory.

Appendix – The Wake of a Fibre

Relative motion between fibre and matrix results in a pyramid of matrix material AZ (fig. 6) being sheared relative to the rest of the matrix, unless diffusion can feed material to the interface at A fast enough to absorb all the relative motion.

If shear occurs as depicted in fig. 6 the relative motion can be imagined to occur through the motion of the edge dislocation loops indicated. The loops will shrink under the united action of line tension and the local tensile stress; their diminishing diameter towards Z is indicated in the figure. No loop therefore exists for longer than a certain time. Consequently, in the wake of each fibre there exists a knowable number of loops, determined by the speed of relative motion between matrix and fibre at A, which governs the rate at which loops are generated, and by the lifetime of a loop.

The loop lifetime can be estimated as follows, where only the driving force from loop energy is taken account of. The energy of a loop of radius r and Burgers vector $\mathbf{b}$ is $w \sim G\mathbf{b}^2 r \ln r/\mathbf{b}$. There are $n = \pi r^2/\mathbf{b}^2$ vacancies of atomic volume $\mathbf{b}^3$ in this loop. Hence

$$\partial w/\partial n = \frac{\partial w}{\partial r} \cdot \frac{\partial r}{\partial n} = \left(\frac{\mathbf{b}^2}{2\pi r}\right)\left[G\mathbf{b}^2 \ln \frac{r}{\mathbf{b}} + G\mathbf{b}^2\right] \sim \frac{G\mathbf{b}^4}{2\pi r} \ln \frac{r}{\mathbf{b}}$$

In shrinking, the loops create a vacancy supersaturation c in the adjacent matrix, which causes vacancies to be transported away at the rate $\mathrm{d}n/\mathrm{d}t = (-4\pi Dr/\mathbf{b}^3)(c/c_0 - 1)$, where D is the diffusion coefficient and spherical symmetry has been assumed to a sink of equilibrium concentration c_0 at infinity [9]. We therefore have

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}n} \cdot \frac{\mathrm{d}n}{\mathrm{d}t} = \frac{\mathbf{b}^2}{2\pi r}\left(-\frac{4\pi Dr}{\mathbf{b}^3}\right)\left(\frac{c}{c_0} - 1\right).$$

Now $c = c_0 \exp(\partial w/\partial n)/kT$. Substituting for c and $\partial w/\partial n$ gives

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{2D}{\mathbf{b}}\left(\exp \frac{G\mathbf{b}^4 \ln r/\mathbf{b}}{2\pi rkT} - 1\right)$$

which, since $G\mathbf{b}^4 \ln r/\mathbf{b} \ll 4\pi rkT$, becomes

$$\frac{\mathrm{d}r}{\mathrm{d}t} \sim -\frac{2D}{\mathbf{b}} \cdot \frac{G\mathbf{b}^4 \ln r/\mathbf{b}}{2\pi rkT} \sim \frac{DG\mathbf{b}^3}{\pi rkT}.$$

Integrating, $r^2/2 \sim -(DG\mathbf{b}^3 t)/\pi kT$, + constant. Since at $t = 0$, $r = a$, the constant $= a^2/2$. Hence $r^2 \sim a^2 - (2DG\mathbf{b}^3 t)/\pi kT$, and the time t' required for the loop to shrink to zero is $t' = (\pi kTa^2)/2DG\mathbf{b}^3$. As an example, for Al at 300°C, $D \sim 1 \times 10^{-10}$ cm^2/sec and $G \sim 2.5 \times 10^{11}$ d/cm^2. Putting $a = 1$ μm gives $t' = 3.7$ sec.

A useful alternative expression for t' is obtained by replacing D with the appropriate function of the viscosity coefficient η. We have $D = kT/6\pi\mathbf{b}\eta$ [10], which gives

$$t' = (3\pi^2 a^2\eta)/G\mathbf{b}^2.$$

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Received 21 September and accepted 28 September 1971.